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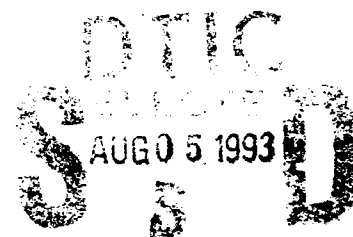
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## OPTIMAL FIXED-FINITE-DIMENSIONAL COMPENSATOR FOR BURGERS' EQUATION WITH UNBOUNDED INPUT/OUTPUT OPERATORS

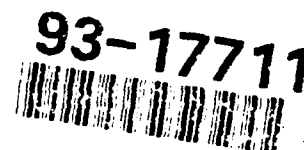
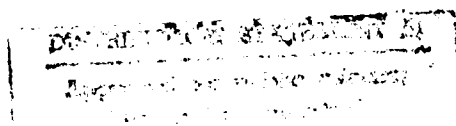
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# Optimal Fixed-Finite-Dimensional Compensator for Burgers' Equation with Unbounded Input/Output Operators

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## ABSTRACT

In this paper we consider the problem of using reduced order dynamic compensators to control a class of nonlinear parabolic distributed parameter systems. We concentrate on a system with unbounded input and output operators governed by Burgers' equation. We use a linearized model to compute low-order-finite-dimensional control laws by minimizing certain energy functionals. We then apply these laws to the nonlinear model. Standard approaches to this problem employ model/controller reduction techniques in conjunction with LQG theory. The approach used here is based on the finite-dimensional Bernstein/Hyland optimal projection theory which yields a fixed-finite-order controller.

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## 1 Introduction

In recent years considerable attention has been devoted to the problem of using feedback to control fluid dynamic systems. This problem is complex and particularly difficult when one is faced with phenomena such as shocks. Moreover, these systems are governed by nonlinear partial differential equations so that the natural state of the system is infinite dimensional. If one assumes that "full state feedback" is necessary to design practical controllers, then one would conclude that feedback control of fluid dynamic system is "not practical". However, it is well known that even in finite dimensional control systems one rarely has the ability to accurately sense all states, so that some form of dynamic compensation must be used.

This idea clearly extends to infinite dimensional problems and there is a growing literature on observers/compensators for distributed parameter systems. In this paper we consider a boundary control problem governed by Burgers' equation. We selected this problem because Burgers' equation is an infinite dimensional model that captures some phenomena (e.g., shocks) often observed in fluid flows and because it is simple enough to provide real insight into the problem. The goal is to show that it is possible to use modern control theory to produce practical finite dimensional dynamic compensators for boundary control of nonlinear partial differential equations of the type that occur naturally in fluid dynamics.

We shall present a short summary of one approach (the optimal projection method due to Bernstein and Hyland) and show how this approach can be used in conjunction with standard numerical schemes to produce a realizable low order controller. The optimal projection method is one of many approaches to this problem. However, we shall concentrate on this method because a very nice theory has already been developed (for bounded input and output operators) and we are more interested in illustrating (to non-experts) that recent results in distributed parameter control theory can be used to design practical feedback laws, than in discussing the "best" approach to the problem. It will be clear from our presentation that we are writing for those that are not necessarily "control experts". The extension of the general theoretical results to unbounded input and output operators will appear in a forthcoming paper. However, for the compensators presented here, we do not need the most general theory since we use the finite dimensional version of the optimal projection method.

As noted above it is almost impossible to observe the whole state. Con-

trols and sensors are limited to a few points or segments of the boundary, so it is necessary to construct an appropriate observer (estimator) of the state and design a feedback control law (called a compensator) based on the information available from the observed (estimated) state variable. Boundary control and observation often leads to unbounded input and output operators. Stabilization by dynamic feedback or compensation has been considered by Curtain [5], Fujii [8], and Nambu [12] for classes of parabolic as well as hyperbolic systems, including control and observation at the boundary. All of these approaches produce stabilization schemes that either have the same finite order as that of a high-order approximate model, or alternatively, open-loop model reduction or closed-loop control reduction techniques are applied to achieve a lower-order compensator. An advance was made by Schumacher [15], when he gave a theory for designing finite-dimensional compensators for a large class of systems, including parabolic and delay systems. However, in his theory it was assumed that the control and observation operators are bounded. Curtain [4] presented an alternative compensator design which applied to the same class of systems, except that unbounded inputs and outputs were allowed. In [14], Pritchard and Salamon established a framework based on semigroup theory for treating the linear quadratic regulator problem for infinite-dimensional systems with unbounded input and output operators. Their approach is based on a weak formulation of the Riccati equations which characterize the optimal feedback law in an appropriate dual space.

Here we consider the problem of designing a fixed-finite-dimensional compensator for a class of distributed system governed by Burgers' equation, where the control and the observation are implemented at the boundary of the domain. The possibility of applying this approach to distributed parameter systems was first suggested by Johnson in [9] and Pearson [13]. The idea of fixing the order of the finite-dimensional compensator, while retaining the distributed parameter model was expanded and developed by Bernstein and Hyland in [1] and [2]. The method extends the full order LQG case to an "optimal fixed-finite-order compensator" characterized by four equations; two modified Riccati equations and two modified Lyapunov equations, coupled by an oblique projection whose rank is precisely equal to the order of the compensator. Bernstein and Hyland assumed that the control and observation operators were bounded and hence boundary control and observations were not covered by their theory.

We will present a Bernstein/Hyland type fixed-finite-dimensional compensator design, which does extend to unbounded input/output problems.

In Section 2 we discuss the existence of a finite-dimensional compensator for parabolic distributed parameter systems with unbounded control and observation. In Section 3 we summarize the infinite-dimensional optimal projection theory from [1], and derive the corresponding equations and feedback gains which characterize the fixed-finite-order compensator. In Section 4 we present an example, construct the approximation schemes, and discuss the computational algorithm used for the optimal projection design synthesis. Finally, Section 5 contains numerical results and Section 6 is devoted to a few closing remarks.

## 2 A Theoretical Existence Result

We consider the following abstract Cauchy problem

$$\dot{z}(t) = Az(t) + Bu(t), \quad z(0) = z_0 \in H \quad (1)$$

$$y(t) = Cz(t) \quad t \geq 0 \quad (2)$$

where  $H$  is a Hilbert space,  $u(\cdot) \in L^2(0, T; \mathfrak{R}^m)$ ,  $y(\cdot) \in L^2(0, T; \mathfrak{R}^l)$ , and  $A$  is the infinitesimal generator of analytic semigroup  $S(t)$  on  $H$ , generally unstable, with exponential growth rate

$$w_0 = \lim_{t \rightarrow \infty} t^{-1} \log \|S(t)\|_{\mathcal{L}(H)} > 0 \quad (3)$$

so that

$$\|S(t)\|_{\mathcal{L}(H)} \leq Me^{(w_0 + \epsilon)t} \quad \text{for all } \epsilon > 0, t \geq 0 \quad (4)$$

for some constant  $M = M(w_0, \epsilon) \geq 1$ . Throughout the remainder of this paper we let  $\hat{A}$  denote the translation  $\hat{A} = -A + wI$ , where  $w$  is fixed and  $w > w_0$ , so that  $\hat{A}$  has well-defined fractional powers  $(\hat{A})^\mu$  on  $H$  and  $-\hat{A}$  is the generator of a strongly continuous analytic semigroup  $\hat{S}(t)$  on  $H$  satisfying

$$\|\hat{S}(t)\|_{\mathcal{L}(H)} \leq \hat{M}e^{-\hat{w}t}, \quad t \geq 0. \quad (5)$$

In order to allow for unbounded operators  $B$  and  $C$ , we assume that  $B \in \mathcal{L}(\mathfrak{R}^m, V)$  and  $C \in \mathcal{L}(W, \mathfrak{R}^l)$ , where  $W$  and  $V$  are also Hilbert spaces such that

$$\mathcal{D}(A) \subseteq W \hookrightarrow H \hookrightarrow V \quad (6)$$

with continuous dense injections. More precisely, we assume that  $B^*$  is

$[\hat{A}^*]^\gamma$ -bounded, or equivalently,

$$[\hat{A}]^{-\gamma} B \in \mathcal{L}(\mathfrak{R}^m, H) \quad \text{for } 0 \leq \gamma < 1. \quad (7)$$

Similarly, for the operator  $C$  we assume that

$$C[\hat{A}]^{-\gamma} \in \mathcal{L}(H, \mathfrak{R}^l) \quad \text{for } 0 \leq \gamma < 1. \quad (8)$$

It is helpful to interpret (1-2) in mild form. In particular, the solution  $z(t)$  is given by

$$z(t) = S(t)z_0 + \int_0^t S(t-s)Bu(s)ds, \quad 0 \leq t \leq T \quad (9)$$

and the output by

$$y(t) = CS(t)z_0 + C \int_0^t S(t-s)Bu(s)ds. \quad (10)$$

We assume that  $S(t)$  is also an analytic semigroup on  $W$  and that the following hypotheses are satisfied:

(H-1) There exists a constant  $b(T) > 0$  such that for every  $T > 0$ ,  $\int_0^T S(T-s)Bu(s)ds \in W$  and

$$\left\| \int_0^T S(T-s)Bu(s)ds \right\|_W \leq b(T) \|u(\cdot)\|_{L^2(0, T; \mathfrak{R}^m)} \quad (11)$$

for every  $u(\cdot) \in L^2(0, T; \mathfrak{R}^m)$ .

(H-2) There exists a constant  $c(T) > 0$  such that for every  $T > 0$ ,

$$\int_0^T \|CS(t)x\|_{L^2(0, T; \mathfrak{R}^l)} dt \leq c(T) \|x\|_V \quad (12)$$

for every  $x \in W$ .

We now give sufficient conditions which imply that the system (1-2) can be stabilized by a finite-dimensional compensator of the form

$$\dot{w}(t) = A_c w(t) - B_c y(t) \quad w(0) = w_0 \quad (13)$$

$$u(t) = C_c w(t) \quad (14)$$

where  $A_c \in \mathbb{R}^{N_c \times N_c}$ ,  $B_c \in \mathbb{R}^{N_c \times m}$ , and  $C_c \in \mathbb{R}^{\ell \times N_c}$  are suitably chosen matrices. We need the following well-posedness result for the connected system (1-2) and (13-14). This result and proof may be found in [4].

**Proposition 2.1** *Let (H-1)-(H-2) be satisfied, then for all  $z_0 \in W$ ,  $w_0 \in \mathbb{R}^{N_c}$  there exists a unique solution pair  $z(t)$  and  $w(t)$  of (1-2) and (13-14). This means that  $z(t)$  is continuous in  $H$  and absolutely continuous in  $V$ , that (1) is satisfied for almost every  $t \geq 0$  where  $u(t)$  is given by (14), and that  $w(t) \in \mathbb{R}^{N_c}$  is continuously differentiable and satisfies (13) where  $y(t)$  is given by (2).*

In addition to hypotheses (H-1) and (H-2), we assume:

**(H-3) Stabilizability Condition (S.C.)**

There exists an operator  $F \in \mathcal{L}(H, \mathbb{R}^m)$  such that  $A_F = A + BF$  generates an analytic semigroup  $S_F(t) = e^{(A+BF)t}$  and  $S_F(t)$  is exponentially stable on  $H$ , i.e.,

$$\|S_F(t)\|_{\mathcal{L}(H)} \leq M_F e^{-w_F t}, \quad \text{for } w_F > 0. \quad (15)$$

**(H-4) Detectability Condition (D.C.)**

There exists an operator  $G \in \mathcal{L}(\mathbb{R}^\ell, H)$  such that  $A_G = A + GC$  generates an analytic semigroup  $S_G(t) = e^{(A+GC)t}$  and  $S_G(t)$  is exponentially stable on  $H$ , i.e.,

$$\|S_G(t)\|_{\mathcal{L}(H)} \leq M_G e^{-w_G t}, \quad \text{for } w_G > 0. \quad (16)$$

**(H-5)** In addition to (H-3) and (H-4) there exists a finite-dimensional subspace  $\mathcal{N} \subset W$ , with  $\dim \mathcal{N} \leq N_c$  such that

(i)  $S_F(t)\mathcal{N} \subset \mathcal{N}$ , for all  $t \geq 0$ ,

(ii)  $\text{Range } G \subset \mathcal{N}$ ,

(iii)  $\mathcal{N} \subseteq \mathcal{D}(A_F)$ .

Moreover, there exist linear maps  $i: \mathbb{R}^{N_c} \rightarrow \mathcal{N}$ ,  $\pi: H \rightarrow \mathbb{R}^{N_c}$  such that

$$\pi i = I_{N_c}, \quad i \pi x = x \quad \text{for } x \in \mathcal{N}. \quad (17)$$

Note that (H-5) implies that  $\pi A_F i$  is a well defined linear map on  $\mathbb{R}^{N_c}$ . We will show that the system

$$\dot{w}(t) = \pi (A_F + GC) i w(t) - \pi G y(t), \quad w(0) = w_0 \quad (18)$$

$$u(t) = F i w(t) \quad (19)$$

defines a stabilizing compensator for the Cauchy problem (1-2). The following result is a slight extension of Theorem 2.5 in [7] for unbounded inputs and outputs.

**Theorem 2.2** *If (H-1)-(H-5) are satisfied, then the closed-loop system defined by (1-2) and (18-19) is exponential stable.*

**Proof:** Note that without loss of generality we can assume that  $\dim \mathfrak{R} = N_c$ . By Proposition 2.1 it follows that the closed-loop system is a well-posed Cauchy problem. Let  $z_0 \in W$ ,  $w_0 \in \mathfrak{R}^{N_c}$  and  $z(t)$ ,  $w(t)$  be defined by (1-2) and (18-19), respectively. Since  $z(t) \in W$ , if  $x(t)$  is defined by

$$x(t) = i w(t) - z(t) \quad t \geq 0,$$

then  $x(t)$  belongs to  $W$  and it is straightforward to show that

$$\dot{w}(t) = \pi A_F i w(t) + \pi G C x(t). \quad (20)$$

Therefore,

$$\begin{aligned} x(t) &= i \pi S_F(t) i w_0 + \int_0^t i \pi S_F(t-s) i \pi G C x(s) ds - z(t) \\ &= S_F(t) i w_0 + \int_0^t S_F(t-s) G C x(s) ds - z(t) \\ &= S(t) i w_0 + \int_0^t S(t-s) [B F i w(s) + G C x(s)] ds \\ &\quad - S(t) z_0 + \int_0^t S(t-s) B u(s) ds \\ &= S(t) x(0) + \int_0^t S(t-s) G C x(s) ds, \end{aligned}$$

which implies that  $x(t) = S_G(t) x(0)$ . The stability of  $x(t)$ ,  $w(t)$  and  $z(t)$  follows.



### 3 Optimal Projection Theory

Consider the steady-state fixed-order dynamic compensator problem, defined by the infinite-dimensional control system

$$\dot{z}(t) = Az(t) + Bu(t) + H_1\eta(t) \quad (21)$$

with measurements

$$y(t) = Cz(t) + H_2\eta(t). \quad (22)$$

The objective is to design a finite-dimensional fixed-order dynamic compensator

$$\dot{z}_c(t) = A_c z_c(t) + B_c y(t) \quad (23)$$

$$u(t) = C_c z_c(t) \quad (24)$$

which minimizes the steady-state performance criterion

$$J(A_c, B_c, C_c) \stackrel{\text{def}}{=} \lim_{t \rightarrow \infty} \frac{1}{t} \int_0^t \mathbf{E} \left[ \langle R_1 z(s), z(s) \rangle + u(s)^T R_2 u(s) \right] ds \quad (25)$$

where the operators  $A$ ,  $B$  and  $C$  satisfy all the assumptions given in the previous section and  $\mathbf{E}[\cdot]$  is the expectation. In addition, assume that the state and measurements are corrupted by a white noise  $\eta(t)$  in the Hilbert space  $\tilde{H}$ , with zero-mean Gaussian,  $H_1 \in \mathcal{L}(\tilde{H}, H)$ ,  $H_2 \in \mathcal{L}(\tilde{H}, \mathbb{R}^\ell)$ ,  $R_1 \in \mathcal{L}(H)$  is self-adjoint and nonnegative definite, and that  $R_2$  is an  $m \times m$  symmetric positive-definite matrix. We assume that the disturbance and measurements are independent, i.e.,  $H_1 H_2^* = 0$ ,  $V_1 = H_1 H_1^* \in \mathcal{L}(H)$  is nonnegative definite and of trace class, and that  $V_2 = H_2 H_2^* \in \mathbb{R}^{\ell \times \ell}$  is positive definite. Also, it is assumed that the initial state  $z(0) = z_0$  is Gaussian and independent of  $\eta(\cdot)$ . The compensator will be assumed to be of fixed, finite order  $N_c$  (i.e.,  $z_c(t) \in \mathbb{R}^{N_c}$ ) and the optimization is performed over  $A_c \in \mathbb{R}^{N_c \times N_c}$ ,  $B_c \in \mathbb{R}^{N_c \times \ell}$  and  $C_c \in \mathbb{R}^{m \times N_c}$ .

If one introduces the augmented state space  $\mathcal{H} = H \times \mathbb{R}^{N_c}$ , then the closed-loop system becomes a linear system on  $\mathcal{H}$ . Consequently, define the closed-loop operator  $\mathcal{A} : \mathcal{D}(\mathcal{A}) \subseteq \mathcal{H} \rightarrow \mathcal{H}$  on the dense domain  $\mathcal{D}(\mathcal{A}) = \mathcal{D}(A) \times \mathbb{R}^{N_c}$  by

$$\mathcal{A} = \begin{bmatrix} A & BC_c \\ B_c C & A_c \end{bmatrix} = \begin{bmatrix} A & 0 \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & BC_c \\ B_c C & A_c \end{bmatrix}.$$

Since the operator

$$\begin{bmatrix} A & 0 \\ 0 & 0 \end{bmatrix} : \mathcal{D}(\mathcal{A}) \rightarrow \mathcal{H},$$

generates an analytic semigroup

$$\begin{bmatrix} e^{At} & 0 \\ 0 & I_{n_c} \end{bmatrix} \quad t \geq 0,$$

then conditions (7)-(8) imply that  $\mathcal{A}$  is also closed and generates an analytic semigroup  $e^{At}$  on  $\mathcal{H}$  (see [10]). To guarantee that  $J$  is finite and independent of initial conditions we restrict our attention to the set of admissible compensators defined by

$$\mathcal{S} = \{(A_c, B_c, C_c) : e^{A_c t} \text{ is exponentially stable}\}. \quad (26)$$

If  $(A_c, B_c, C_c) \in \mathcal{S}$ , then there exist  $\alpha \geq 1$  and  $\beta > 0$  such that

$$\|e^{A_c t}\| \leq \alpha e^{-\beta t} \quad t \geq 0. \quad (27)$$

Moreover, we know from Theorem 2.2 above that  $\mathcal{S}$  is non-empty. We now state some results found in [1] and [2].

**Lemma 3.1** *If  $\hat{Q}$  and  $\hat{P} \in \mathcal{L}(H)$  have finite rank and are nonnegative definite, then  $\hat{Q}\hat{P}$  is nonnegative semi-simple. Furthermore, if  $\text{rank}(\hat{Q}\hat{P}) = N_c$ , then there exist  $G$  and  $\Gamma \in \mathcal{L}(H, \mathbb{R}^{N_c})$  and a positive semi-simple matrix  $M \in \mathbb{R}^{N_c \times N_c}$  such that*

$$\hat{Q}\hat{P} = G^* M \Gamma \quad (28)$$

$$\Gamma G^* = I_{N_c}. \quad (29)$$

**Proof:** Bernstein and Hyland give a complete proof of this result in [1]. Here we outline their proof in order to illustrate the form of the factorization of  $\hat{Q}\hat{P}$  and to provide a description of the operators  $G$  and  $\Gamma$ . Since  $\hat{Q}$  and  $\hat{P}$  have finite rank, there exists a finite dimensional subspace  $\mathcal{Z} \subset H$  such that  $\hat{Q}\mathcal{Z} \subset \mathcal{Z}$ ,  $\hat{Q}\mathcal{Z}^\perp = 0$ ,  $\hat{P}\mathcal{Z} \subset \mathcal{Z}$  and  $\hat{P}\mathcal{Z}^\perp = 0$ . Hence there exists an orthonormal basis for  $H$  and in this basis  $\hat{Q}$  and  $\hat{P}$  have the infinite matrix representations

$$\hat{Q} = \begin{bmatrix} \hat{Q}_1 & 0 \\ 0 & 0 \end{bmatrix}, \quad \hat{P} = \begin{bmatrix} \hat{P}_1 & 0 \\ 0 & 0 \end{bmatrix},$$

where  $\hat{Q}_1, \hat{P}_1 \in \mathfrak{R}^{r \times r}$  and  $r = \dim \mathcal{Z}$ . Consequently, there exists an invertible  $\Phi \in \mathfrak{R}^{r \times r}$  such that  $\hat{\Lambda} = \Phi^{-1} \hat{Q}_1 \hat{P}_1 \Phi$  is nonnegative and diagonal and  $\hat{Q} \hat{P}$  is nonnegative and semi-simple. If  $\text{rank}(\hat{Q} \hat{P}) = N_c$ , then it is clear that  $\Phi$  can be chosen so that

$$\hat{\Lambda} = \begin{bmatrix} \Lambda & 0 \\ 0 & 0 \end{bmatrix}$$

where  $\Lambda \in \mathfrak{R}^{N_c \times N_c}$  is positive and diagonal. Hence,

$$\hat{Q} \hat{P} = \begin{bmatrix} \Phi & 0 \\ 0 & I_\infty \end{bmatrix} \begin{bmatrix} I_{N_c} \\ 0 \\ 0 \end{bmatrix} \Lambda \begin{bmatrix} I_{N_c} & 0 & 0 \end{bmatrix} \begin{bmatrix} \Phi^{-1} & 0 \\ 0 & I_\infty \end{bmatrix},$$

and if we define  $G, M$  and  $\Gamma$  by

$$\begin{aligned} G &= \begin{bmatrix} S^\top & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \Phi^\top & 0 \\ 0 & I_\infty \end{bmatrix} \\ \Gamma &= \begin{bmatrix} S^{-1} & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \Phi^{-1} & 0 \\ 0 & I_\infty \end{bmatrix} \\ M &= S^{-1} \Lambda S. \end{aligned}$$

for any invertible  $S \in \mathfrak{R}^{N_c \times N_c}$ , then  $G, \Gamma$  and  $M$  provide the desired factorization and this completes the proof.

Throughout the paper we will refer to  $G, \Gamma$  and  $M$  satisfying the above lemma as a  $(G - M - \Gamma)$ -factorization of  $\hat{Q} \hat{P}$ . For convenience we define  $\Sigma = BR_2^{-1}B^*$  and  $\bar{\Sigma} = C^*V_2^{-1}C$  and let  $I_{N_c}$  and  $I_H$  denote respectively the  $N_c \times N_c$  identity matrix and the identity operator on  $H$ , respectively. We state Bernstein's and Hyland's main theorem which provides a set of necessary conditions that characterize the optimal steady-state fixed order dynamic compensator for bounded input and output operators (see [1]).

**Theorem 3.2** *Let  $B$  and  $C$  be bounded operators and let  $N_c$  be given and suppose that there exists a controllable and observable dynamic compensator  $(A_c, B_c, C_c) \in \mathcal{S}$  of order  $N_c$  which minimizes  $J$  given by (25), then there exist nonnegative definite operators  $Q, P, \hat{Q}$ , and  $\hat{P}$  on  $H$  such that  $A_c, B_c$ , and  $C_c$  are given by*

$$A_c = \Gamma(A - Q\bar{\Sigma} - \Sigma P)G^* \quad (30)$$

$$B_c = \Gamma Q C^* V_2^{-1} \quad (31)$$

$$C_c = -R_2^{-1} B^* P \Gamma^* \quad (32)$$

for some  $(G - M - \Gamma)$ -factorization of  $\hat{Q}\hat{P}$  and such that, with  $\tau = G^* \Gamma \in \mathcal{L}(H)$ , the following conditions are satisfied:

$$\begin{aligned} Q &: \mathcal{D}(A^*) \rightarrow \mathcal{D}(A) & P &: \mathcal{D}(A) \rightarrow \mathcal{D}(A^*) \\ \hat{Q} &: H \rightarrow \mathcal{D}(A) & \hat{P} &: H \rightarrow \mathcal{D}(A^*) \\ \text{rank}(\hat{Q}) &= \text{rank}(\hat{P}) = \text{rank}(\hat{Q}\hat{P}) \end{aligned}$$

and

$$0 = (A - \tau Q \bar{\Sigma})Q + Q(A - \tau Q \bar{\Sigma})^* + V_1 + \tau Q \bar{\Sigma} Q \tau^* \quad (33)$$

$$0 = (A - \Sigma P \tau)^* P + P(A - \Sigma P \tau) + R_1 + \tau^* P \Sigma P \tau \quad (34)$$

$$0 = [(A - \Sigma P)\hat{Q} + \hat{Q}(A - \Sigma P)^* + Q \bar{\Sigma} Q] \tau^* \quad (35)$$

$$0 = [(A - Q \bar{\Sigma})^* \hat{P} + \hat{P}(A - Q \bar{\Sigma}) + P \Sigma P] \tau. \quad (36)$$

Note that these necessary conditions consist of a system of four operator equations, including a pair of modified Riccati equations and a pair of modified Lyapunov equations which are coupled by the operator  $\tau \in \mathcal{L}(H)$ . The operator  $\tau$  is idempotent, since  $\tau^2 = \tau \tau = G^* \Gamma G^* \Gamma = G^* I_{n_c} \Gamma = G^* \Gamma = \tau$ . In general  $\tau$  is an oblique projection and may not be orthogonal since there is no requirement that  $\tau$  be self-adjoint. Moreover, we note that in view of Lemma 3.1, Theorem 3.2 applies to  $(SA_c S^{-1}, SB_c, C_c S^{-1})$  for any invertible  $S \in \mathfrak{R}^{N_c \times N_c}$ , since the  $(G - M - \Gamma)$ -factorization of  $\hat{Q}\hat{P}$ , used to determine  $A_c$ ,  $B_c$  and  $C_c$ , is not unique. However, the operator  $\tau$  remains invariant over the class of factorizations. An easy computation yields the following identities:

$$\hat{Q} = \tau \hat{Q} \quad \text{and} \quad \hat{P} = \hat{P} \tau. \quad (37)$$

It is helpful to have an alternative form of the optimal projection equations to actually compute the optimal fixed-order compensator of the approximating finite-dimensional plant. The following result for bounded input bounded output operators may be found in [1].

**Proposition 3.1** *If  $B$  and  $C$  are bounded, then the optimal projection equations (33)–(36) are equivalent, respectively, to*

$$0 = A Q + Q A^* + V_1 - Q \bar{\Sigma} Q + \tau_1 Q \bar{\Sigma} Q \tau_1^* \quad (38)$$

$$0 = A^*P + PA + R_1 - P\Sigma P + \tau_{\perp}^* P\Sigma P \tau_{\perp} \quad (39)$$

$$0 = A_p \hat{Q} + \hat{Q} A_p + Q \bar{\Sigma} Q - \tau_{\perp} Q \bar{\Sigma} Q \tau_{\perp}^* \quad (40)$$

$$0 = A_q^* \hat{P} + \hat{P} A_q + P\Sigma P - \tau_{\perp}^* P\Sigma P \tau_{\perp} \quad (41)$$

where

$$\tau_{\perp} = I_H - \tau, \quad A_p = A - \Sigma P \quad \text{and} \quad A_q = A - Q \bar{\Sigma}. \quad (42)$$

This form of the optimal projection equations shows that there is a connection between Theorem 3.2 and the standard LQG result when  $\dim H = N < \infty$ . In this case, we note that the  $(G - M - \Gamma)$ -factorization of  $\hat{Q}\hat{P}$  when  $N_c = N$  is given by  $G = \Gamma = I_N$  and  $M = \hat{Q}\hat{P}$ . Since  $\tau = I_N$  and  $\tau_{\perp} = 0$ , it follows that (38)-(39) reduce to the standard observer and regulator Riccati equations.

To obtain a geometric interpretation of the optimal projection we introduce the "quasi-full-state" estimate

$$\hat{z}(t) = G^* z_c(t) \in H, \quad (43)$$

so that  $\tau \hat{z}(t) = \hat{z}(t)$  and  $z_c(t) = \Gamma \hat{z}(t)$ . Hence, the closed-loop system can be written as

$$\dot{z}(t) = Az(t) + B\hat{C}_c \tau \hat{z}(t) \quad (44)$$

$$\dot{\hat{z}}(t) = \tau(A + B\hat{C}_c - \hat{B}_c C) \tau \hat{z}(t) + \tau \hat{B}_c C z(t) \quad (45)$$

where

$$\hat{B}_c = QC^*V_2^{-1} \quad \text{and} \quad \hat{C}_c = -R_2^{-1}B^*P. \quad (46)$$

This shows that the geometric structure of the quasi-full-order compensator is dictated by the projection  $\tau$ . Sensor inputs  $\tau \hat{B}_c C z$  are annihilated unless they are contained in  $\mathcal{R}(\tau^*) = \mathcal{N}(\tau)^{\perp}$ , while  $\tau \hat{z}$  employed in the control input is contained in  $\mathcal{R}(\tau)$ . Consequently,  $\mathcal{R}(\tau)$  and  $\mathcal{R}(\tau^*)$  are the control and observation subspaces of the compensator, respectively. In order to modify the previous results so that they will apply directly to unbounded  $B$  and  $C$  operators, care must be exercised to precisely define the weak forms of (33)-(36) and (38)-(41). We shall not consider this problem in this short note. However, we shall use these systems to guide the approximations below.

## 4 Finite Dimensional Approximation

In general, the optimal projection equations (38)–(41) are infinite dimensional operator equations. To actually use these equations to compute the optimal fixed-finite-order compensator, a finite dimensional approximation is needed (see [2] for details).

Let  $H^N$  for  $N = 1, 2, \dots$ , be a sequence of finite dimensional linear subspaces of  $H$  and let  $\mathcal{P}^N : H \rightarrow H^N$  be the canonical orthogonal projections. Let  $A^N \in \mathcal{L}(H^N)$ ,  $B^N \in \mathcal{L}(\mathfrak{R}^m, H^N)$ ,  $C^N \in \mathcal{L}(H^N, \mathfrak{R}^l)$ ,  $R_1^N \in \mathcal{L}(H^N)$  and  $V_1^N \in \mathcal{L}(H^N)$  be given and consider the approximating system

$$\dot{z}^N(t) = A^N z^N(t) + B^N u^N(t) + H_1^N \eta^N(t) \quad (47)$$

$$y^N(t) = C^N z^N(t) + H_2^N \eta^N(t). \quad (48)$$

The goal is to design a sequence of finite-dimensional dynamic compensators of fixed order  $N_c$  of the form

$$\dot{z}_c^N(t) = A_c^N z_c^N(t) + B_c^N y^N(t) \quad (49)$$

$$u^N(t) = C_c^N z_c^N(t), \quad (50)$$

which minimizes the performance criterion

$$J^N(A_c^N, B_c^N, C_c^N) \stackrel{\text{def}}{=} \lim_{t \rightarrow \infty} \frac{1}{t} \int_0^t \mathbf{E} \left[ \langle R_1^N z^N(s), z^N(s) \rangle + u(s)^T R_2 u(s) \right] ds. \quad (51)$$

Now, for each  $N = 1, 2, \dots$ , let  $\{\phi_j^N\}_{j=1}^{k^N}$  be a basis for  $H^N$ . Also, for any linear operator  $F^N$  with domain and range in  $H^N$ , unless otherwise noted, we use the same symbol  $F^N$  for its matrix representation with respect to the basis chosen. Let  $\Psi^N$  denote the  $k^N$ -square Gram matrix corresponding to the basis  $\{\phi_j^N\}_{j=1}^{k^N}$  (e.g.,  $\Psi^N = [\langle \phi_i^N, \phi_j^N \rangle_{H^N}]$ ). Note that

$$(A^N)^* = (\Psi^N)^{-1} (A^N)^T \Psi^N \quad (B^N)^* = (B^N)^T \Psi^N \quad (52)$$

$$(C^N)^* = (\Psi^N)^{-1} (C^N)^T \quad (\Sigma^N) = B^N R_2^{-1} (B^N)^T \Psi^N \quad (53)$$

$$(\tau_\perp^N)^* = (\Psi^N)^{-1} (\tau_\perp^N)^T \Psi^N \quad (\bar{\Sigma}^N) = (\Psi^N)^{-1} (C^N)^T V_2^{-1} C^N \quad (54)$$

and if we define the  $k^N \times k^N$  nonnegative definite matrices

$$\begin{aligned} Q_0^N &\stackrel{\text{def}}{=} Q^N(\Psi^N)^{-1} & P_0^N &\stackrel{\text{def}}{=} \Psi^N P^N \\ \hat{Q}_0^N &\stackrel{\text{def}}{=} \hat{Q}^N(\Psi^N)^{-1} & \hat{P}_0^N &\stackrel{\text{def}}{=} \Psi^N \hat{P}^N \\ V_0^N &\stackrel{\text{def}}{=} V_1^N(\Psi^N)^{-1} & R_0^N &\stackrel{\text{def}}{=} \Psi^N R_1^N \\ \Sigma_0^N &\stackrel{\text{def}}{=} B^N R_2^{-1} (B^N)^\top & \bar{\Sigma}_0^N &\stackrel{\text{def}}{=} (C^N)^\top V_2^{-1} C^N, \end{aligned}$$

then the matrix equivalence of the operator equation (38)–(41) become

$$0 = A^N Q_0^N + Q_0^N (A^N)^\top + V_0^N - Q_0^N \bar{\Sigma}_0^N Q_0^N + \tau_\perp^N Q_0^N \bar{\Sigma}_0^N Q_0^N (\tau_\perp^N)^\top \quad (55)$$

$$0 = (A^N)^\top P_0^N + P_0^N A^N + R_0^N - P_0^N \Sigma_0^N P_0^N + (\tau_\perp^N)^\top P_0^N \Sigma_0^N P_0^N \tau_\perp^N \quad (56)$$

$$0 = A_{p_o}^N \hat{Q}_0^N + \hat{Q}_0^N (A_{p_o}^N)^\top + Q_0^N \bar{\Sigma}_0^N Q_0^N - \tau_\perp^N Q_0^N \bar{\Sigma}_0^N Q_0^N (\tau_\perp^N)^\top \quad (57)$$

$$0 = (A_{q_o}^N)^\top \hat{P}_0^N + \hat{P}_0^N A_{q_o}^N + P_0^N \Sigma_0^N P_0^N - (\tau_\perp^N)^\top P_0^N \Sigma_0^N P_0^N \tau_\perp^N. \quad (58)$$

The approximating optimal dynamic compensator  $(A_c^N, B_c^N, C_c^N)$  of order  $N_c$  is then given by

$$A_c^N = \Gamma_0^N (A^N - Q_0^N \bar{\Sigma}_0^N - \Sigma_0^N P_0^N) (G_0^N)^\top \quad (59)$$

$$B_c^N = \Gamma_0^N Q_0^N (C^N)^\top V_2^{-1} \quad (60)$$

$$C_c^N = -R_2^{-1} (B^N)^\top P_0^N (\Gamma_0^N)^\top \quad (61)$$

where  $\Gamma_0^N, G_0^N \in \mathfrak{R}^{N_c \times k^N}$  and  $M_0^N \in \mathfrak{R}^{N_c \times N_c}$  provide a  $(G_0^N - M_0^N - \Gamma_0^N)$ -factorization of  $\hat{Q}_0^N \hat{P}_0^N$ .

We turn now to an example. Consider Burgers' equation, with Neumann boundary control given by

$$\frac{\partial}{\partial t} z(t, x) = \epsilon \frac{\partial^2}{\partial x^2} z(t, x) - z(t, x) \frac{\partial}{\partial x} z(t, x), \quad 0 < x < 1, \quad t > 0 \quad (62)$$

$$z(0, x) = z_o(x) \quad (63)$$

$$\frac{\partial}{\partial x} z(t, 0) = -u_1(t), \quad \frac{\partial}{\partial x} z(t, 1) = u_2(t), \quad (64)$$

and observations

$$y_1(t) = z(t, 0) \quad (65)$$

$$y_2(t) = z(t, 1), \quad (66)$$

where  $\epsilon = \frac{1}{\text{Re}} > 0$  and  $\text{Re}$  is the Reynolds number. Initially, we consider the linearized Neumann boundary control problem

$$\frac{\partial}{\partial t} z(t, x) = \epsilon \frac{\partial^2}{\partial x^2} z(t, x), \quad 0 < x < 1, \quad t > 0 \quad (67)$$

$$z(0, x) = z_0(x) \quad (68)$$

$$\frac{\partial}{\partial x} z(t, 0) = -u_1(t), \quad \frac{\partial}{\partial x} z(t, 1) = u_2(t). \quad (69)$$

We will apply the linearized feedback control laws constructed from this model to the nonlinear Burgers' equation. System (67)-(69) can be placed into the standard state space framework by defining the operator  $A_\epsilon$  on  $H = L_2(0, 1)$  by

$$A_\epsilon \phi = \epsilon \phi'' \quad (70)$$

for all  $\phi \in \mathcal{D}(A_\epsilon) = \{\phi \in H^2(0, 1) : \phi'(0) = \phi'(1) = 0\}$ . Define  $W = V^* = H^{\frac{3}{2}}(0, 1) = \mathcal{D}(\hat{A}_\epsilon^{\frac{3}{2}})$  and let  $B : \mathfrak{R}^2 \rightarrow V$  be defined by  $B = \hat{A} \mathcal{N}$  where  $\hat{A} = -A_\epsilon + wI$  and we assume that  $w$  is not an eigenvalue of  $A_\epsilon$  with homogeneous Neumann boundary conditions, so that  $\hat{A}$  is boundedly invertible on  $L_2(0, 1)$ . The Neumann map  $\mathcal{N}$  is defined by the boundary system given in [11, pages 53-56]. Let  $C : W \rightarrow \mathfrak{R}^2$  defined by

$$C\phi = \begin{bmatrix} \phi(0) \\ \phi(1) \end{bmatrix}, \quad (71)$$

The boundary control problem (67)-(69) can be represented by a differential equation

$$\frac{d}{dt} z = A_\epsilon z(t) + Bu, \quad z(0) = z_0 \quad (72)$$

$$y(t) = Cz(t). \quad (73)$$

It is well known that  $A_\epsilon$  generates an analytic semigroup  $S(t)$  on  $H$ . Moreover, the spectrum  $\sigma(A_\epsilon)$  of  $A_\epsilon$  consists of all eigenvalues  $\lambda_n$ ,  $n = 0, 1, 2, \dots$  given by  $\lambda_n = -\epsilon n^2 \pi^2$  and for each eigenvalue  $\lambda_n$  the corresponding eigen-



function  $\phi_n$  is given by

$$\phi_0(x) = 1 \quad \phi_n(x) = \sqrt{2} \cos(n\pi x), \quad 0 < x < 1. \quad (74)$$

One can easily verify conditions (7)-(8), by taking  $\gamma = \frac{1}{4}$  (see [11]). It is straightforward to show that (H-1)-(H-5) are valid.

## 5 Numerical Results

Now, we formulate a specific approximation scheme for the boundary control problem (74). For each  $N = 2, 3, \dots$  let divide the unit interval  $[0, 1]$  into  $N$  equal subinterval  $[x_i, x_{i+1}]$ ,  $x_i = \frac{i-1}{N+1}$ ,  $i = 1, 2, \dots, N+1$ . Let  $H^N = \text{Span} \{h_i^N\}_{i=0}^{N+1}$  where  $h_i^N(\cdot)$  are the standard hat functions defining continuous piecewise linear splines (see [3]). Note that  $k^N = \dim H^N = N+2$  and let the approximate solution  $z^N(t, x)$  of  $z(t, x)$  for equation (72)-(73) be given by

$$z^N(t, x) = \sum_{i=0}^{N+1} z_i^N(t) h_i^N(x) \quad (75)$$

for some  $z_i^N(t) \in \mathfrak{R}$ ,  $i = 0, 1, \dots, N+1$ . Standard finite element approximations yield the ODE system

$$\frac{d}{dt} z^N(t) = A_\epsilon^N z^N(t) + B^N u(t), \quad z^N(0) = z_0^N \quad (76)$$

$$y^N(t) = C^N z^N(t) \quad (77)$$

where the matrices  $A_\epsilon^N$ ,  $B^N$ ,  $C^N$  can be easily computed by using the Ritz-Galerkin approximation.

For our numerical example, we set  $\alpha = \frac{1}{60}$ , the initial condition  $z_0(x) = \sin(\pi x)$ ,  $r_1 = v_1 = 1$  and  $r_2 = v_2 = 10^{-3}$ . Also,  $R_1 = r_1 I_H$ ,  $R_2 = r_2 I_m$ ,  $V_1 = v_1 I_H$ , and  $V_2 = v_2 I_\ell$ . Therefore, it follows from Section 3 that  $R_o^N = r_1 \Psi^N$  and  $V_o^N = v_1 (\Psi^N)^{-1}$  where  $\Psi^N$  is the Gram matrix. In this numerical example we will compare the approximating optimal LQG (i.e.,  $N_c = N+2$ ) with the dynamic compensators of various order  $N_c$ . The optimal projection equations (55)-(58) were solved using the homotopic continuation algorithm described in [16]. The approximating controllers defined by the linear fixed-order compensator ( $B_c^N$  and  $C_c^N$ ) were applied to Burgers' equation (62)-(66).

We note that  $N = 32$  produces converged optimal LQG designs. Hence, the reduced order compensators were tested on both the linear and nonlinear problem using the  $N = 32$  order finite element model.

In the full order case  $N = 32$  and  $N_c = 34$ , the converged feedback and observer functional gains are given in Figures 1 and 2, respectively. Since we are controlling the flux at each end point  $x = 0$  and  $x = 1$ , we have two feedback functional gains, the one plotted with solid line is the flux control gain at the origin and the one plotted with dashed line is the flux control at the end point  $x = 1$ . Similarly, since we are sensing the flow at the origin and the end point, we have two observer gains (solid line for observer gain at the origin and dashed line for the observer gain at  $x = 1$ ). Next, we applied the full order controller to Burgers' equation resulting in the nonlinear closed-loop trajectory given in Figure 3.

In the fixed-order case, we considered the accuracy of the impulse and step responses of the various reduced order compensator designs compared to the corresponding responses of the full order LQG design. Figure 4 illustrates the linear closed-loop impulse response for the full-order LQG and reduced order compensator (of order  $N_c = 16$ ) designs. The impulse response of the linear closed-loop system for the 16th-order compensator is in perfect agreement with the LQG response. Note that in Figure 4 we see only one plot for both designs because both plots are essentially the same. Similar trends are seen (Figure 5) in the comparisons of the step responses (for the same design case) with the corresponding LQG responses.

For the nonlinear closed-loop response, the 16th-order compensator was applied to Burgers' equation and we see (in Figure 6) excellent agreement with the full order closed-loop trajectory response. Hence, replacing the 32nd-order optimal LQG controller by a 16th-order compensator produces a closed-loop system with minor performance degradation.

We also compared the performances of the closed-loop system of the 4th-order compensator with the full order LQG responses. Figures 7 and 8 are the impulse and step responses of the linear closed-loop system, respectively. If one compares these responses with the corresponding responses for the full order LQG controller shown in Figures 4 and 5, then it is clear that the 4th-order compensator performs almost as well as the full order LQG controller. Similar comments hold for the nonlinear closed-loop responses. For example, the 4th-order compensator response (Figure 9, solid line) compares well to the LQG response (Figure 9, Dashed lines), especially after time  $T = 1.0$ .

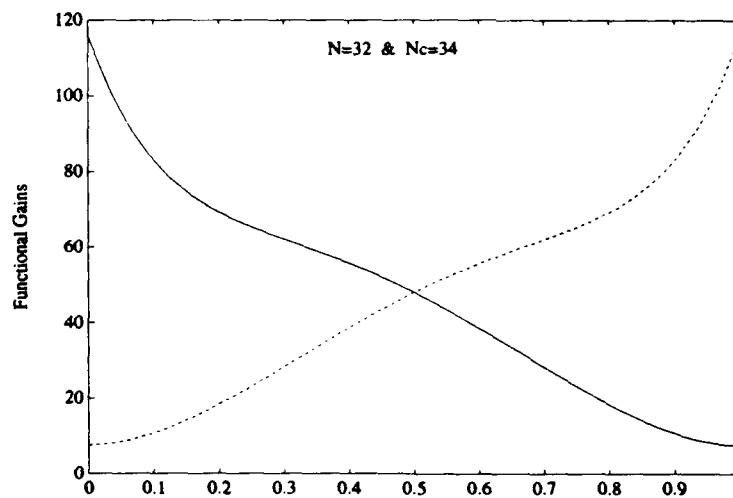


Figure 1: Functional Control Gains for the LQG Case,  $N=32$

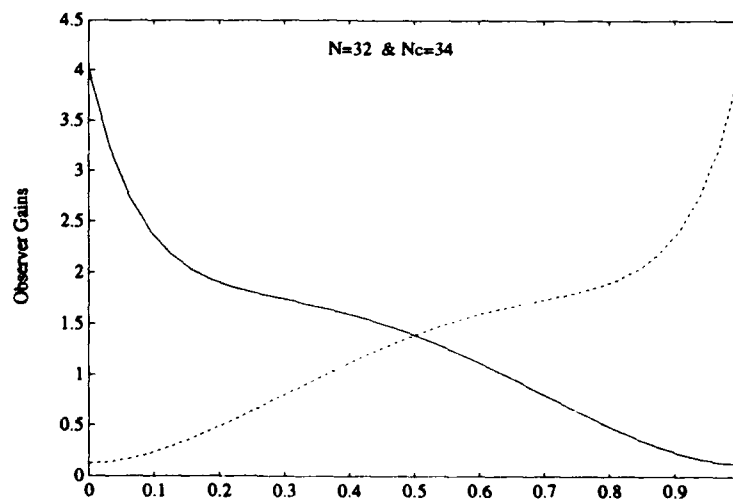


Figure 2: Functional Observer Gains for the LQG Case,  $N=32$

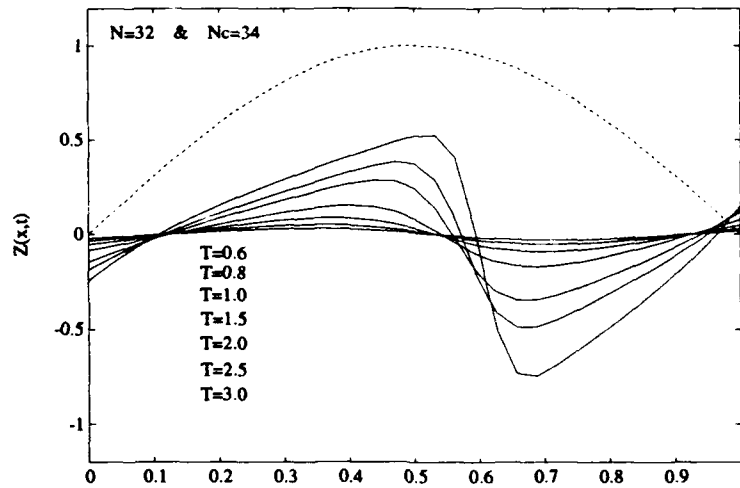


Figure 3: Closed-Loop Trajectory for the LQG Case,  $N=32$

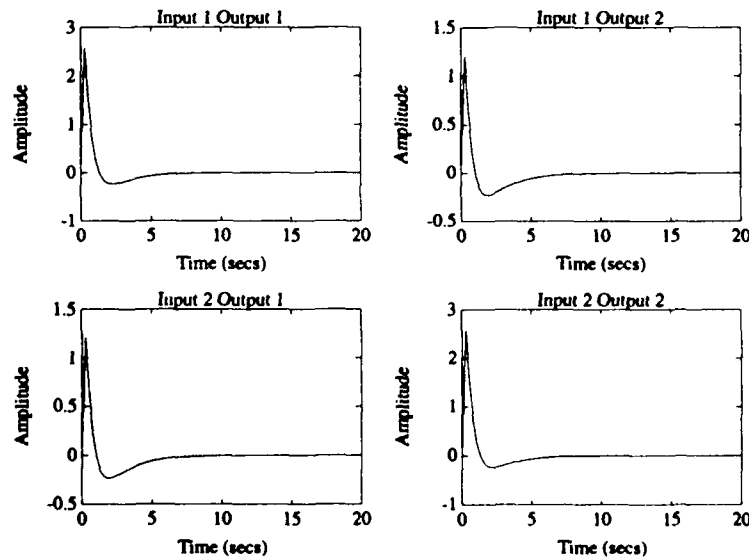


Figure 4: Impulse Response for the LQG & Fixed-Order Cases,  $N=32$ ,  $N_c=16$

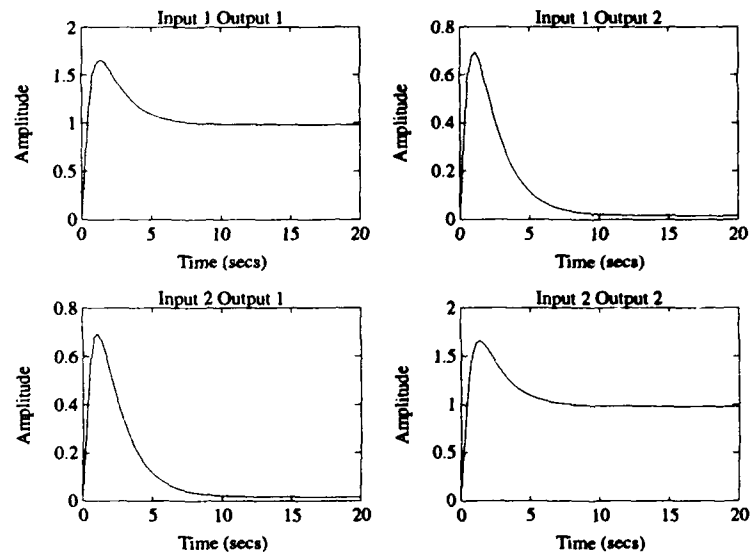


Figure 5: Step Response for the LQG & Fixed-Order Cases,  $N=32$ ,  $N_c=16$

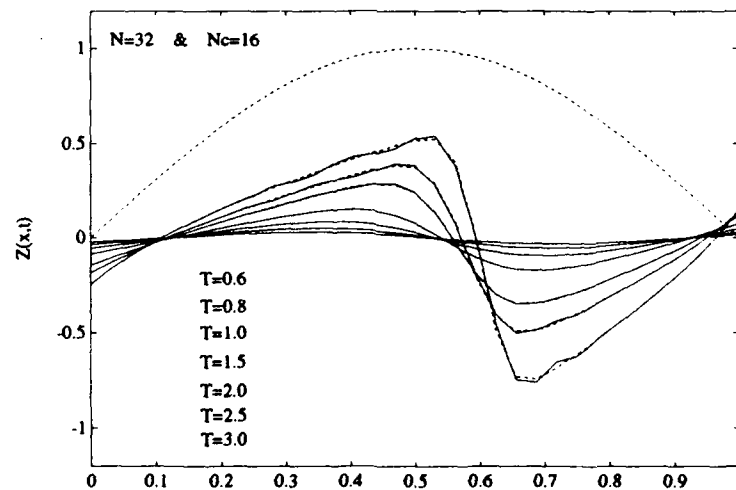


Figure 6: Closed-Loop for the LQG & Fixed-Order Cases,  $N=32$ ,  $N_c=16$

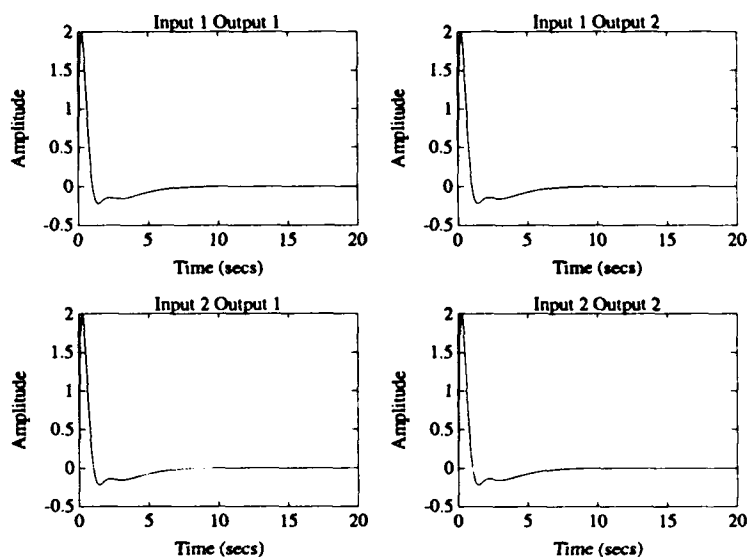


Figure 7: Impulse Response for the Fixed-Order Case,  $N=32$ ,  $N_c=4$

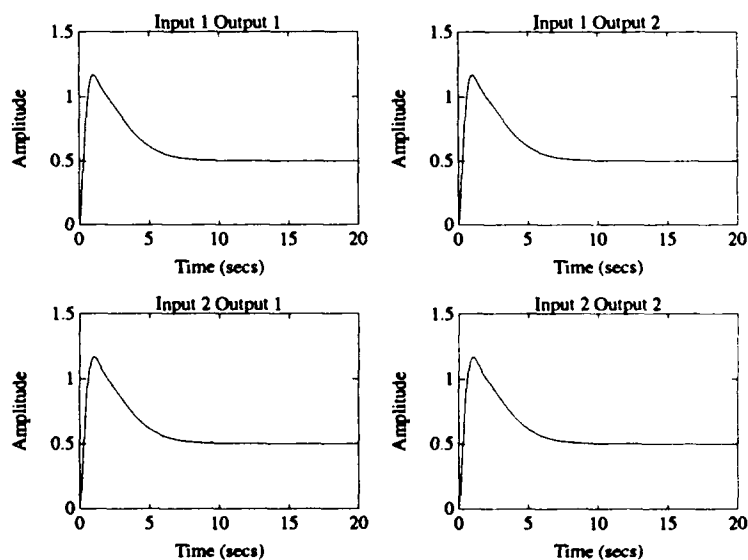


Figure 8: Step Response for the Fixed-Order Case,  $N=32$ ,  $N_c=4$

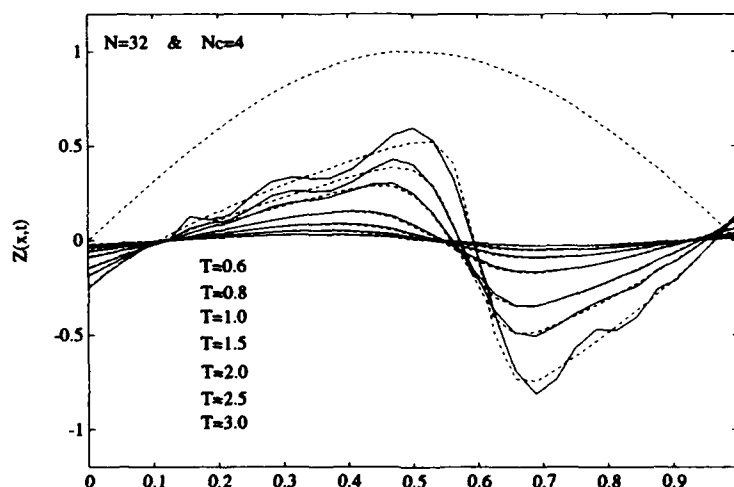


Figure 9: Closed-Loop for the LQG & Fixed-Order Cases,  $N=32$ ,  $N_c=4$

## 6 Conclusion

The purpose of this note was to show that finite dimensional dynamic compensators could be used to control a nonlinear partial differential equation without significant loss in performance. Although there is considerable theoretical and numerical work for bounded input and bounded output operators, numerical results for the unbounded control and observation operators are not as fully developed as the theory. For example, several authors have considered questions of existence of stabilizing dynamic compensators (even for nonlinear plants [6]) for boundary control problems. However, approaches, such as the optimal projection method, that result in a "computable" fixed order compensator have not been applied to more general boundary control problems. Although the numerical results presented here show that the optimal projection method can produce excellent designs for problems with boundary control and observation, there are a number of theoretical and numerical issues that need to be resolved in order to extend this approach to practical problems of this type.

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